Stability Analysis of Multilayer Sandwich Structures

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Nomenclature

$E_{xi}, E_{yi}, \nu_{xi}, \nu_{yi}, E_{xyi}$ = orthotropic elastic constants of ith stiff layer in x- and y-directions
G_{xzj}, G_{yzj} = orthotropic elastic constants of jth core layer in xz- and yz-plane, respectively
w, θ_x, θ_y = vertical displacement, rotations about x- and y-axis, respectively
t_i, h_j = thickness of ith layer and jth core, respectively
σ = stress
$[K]$ = assembled stiffness matrix of a structure
$[K_s]$ = assembled stability matrix of a structure
$\{\delta\}$ = displacement vector
λ = a scaling factor relating to critical stress

Theme

THE finite element method is applied to the stability analysis of multilayer sandwich structures with n stiff layers and $n-1$ weak cores, in which each layer or core can assume different material properties, so that the concept of common shear angle is no longer applicable. A beam element, a rectangular plate element and a triangular element are used in the analysis and their stability matrices are developed. The elements are applied to several standard stability problems and good accuracy is demonstrated. The complex problem of rectangular plate with a circular hole is included to show the versatility of the method.

Contents

The finite element analysis of multilayer sandwich structures, unlike that of conventional sandwich construction has received scant attention. Lundgren and Salama[1] developed a hybrid rectangular element and studied the stability problem of elastic plates. They used the multilayer sandwich plate theory given by Liaw and Little,[2] in which the stiff layers are treated as membranes and a common shear angle is adopted for all the core layers. However, it has been pointed out by Kao and Ross[3] that the theory is not valid for multilayer sandwich beams in which the shear strengths of the individual core layers are different.

Recently, Khatua and Cheung have developed a beam element, a rectangular element[4] and a triangular element[5] of the displacement type to analyze the bending and vibration of beams and plates. In this analysis the idea of common shear angle is excluded by prescribing arbitrary in-plane displacement at different stiff layers. Furthermore, the bending rigidities of stiff layers and orthotropy of the material were taken into account. The same theory is extended herein to study the stability behavior of beams and plates.

Mathematical Formulation: A. General theory. The assumptions of the theory and stress-strain relationships are given previously by the authors[4,5] and will not be repeated here.

B. Finite element analysis. The displacement shape functions, stiffness and mass matrices for beam, rectangular and triangular plate elements have also been given by the authors.[4,5] However, the slope matrix required for computing stability matrix for beam and plate elements will be described here in detail.

B1. Beam element. The strain terms of a stiff layer for computing the stability matrix can be expressed as follows:

$$(w)_{,x} = [G]\{\delta^*\}^e \tag{1}$$

where $\{\delta^*\}^e$ contains the displacement parameters of the bending part only.

B2. Rectangular and triangular plate element. To calculate the stability matrix it is necessary to use the following strain-displacement relationship for a stiff layer

$$\begin{Bmatrix} (w)_{,x} \\ (w)_{,y} \end{Bmatrix} = [G]\{\delta^*\}^e \tag{2}$$

C. Stability matrix for elements. The stability matrix for the elements can be computed from the well known formula[6]

$$[K_s^*]^e = \sum_{i=1}^{n} \int_A t_i [G]^T [\sigma]_i [G] \, dA \tag{3}$$

where * represents the nonzero part of the element stability matrix $[K_s]^e$. The full matrix $[K_s]^e$ and the calculation of $[\sigma]_i$ can be found in the full paper.

Once the element stiffness[4,5] and stability matrices are determined, the stability analysis of a complete structure can be carried out using the following relationship:

$$[K] + \lambda [K_s]\{\delta\} = 0 \tag{4}$$

Numerical Examples. From Eq. (4) it can be seen that linear stability analysis is simply an eigenvalue problem. However, the direct eigenvalue solutions of multilayer sandwich plate by finite element method are often intractable on even the largest of available computers because of the very large matrices involved. In such situations, a technique for eliminating some of the variables is needed so as to reduce the number of degrees of freedom to a manageable size, and the present examples are solved with elimination of all u and v displacements.

A. Beam element. To test the applicability of the theory and the rate of convergence, a simply supported three-layer beam was analyzed. The dimensions and properties are length = 20 in., $t = 0.02$ in., $E = 10^7$ psi, $h = 0.6$ in., and $G = 5000$ psi. The critical stress was calculated to be 739 psi and 734 psi for 4 and 8 mesh divisions in half of the beam, respectively. The theoretical result given by Allen[7] is 731 psi.

B. Rectangular and triangular plate elements. 1) In order to demonstrate the accuracy and convergence of plate elements,

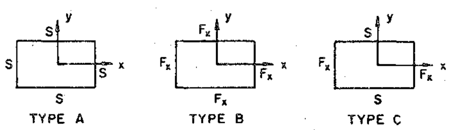

Fig. 1 Description of boundary conditions.

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Table 1 Critical stresses for three-layered plates uniaxially loaded in the x-direction

Aspect ratio	Boundary conditions (see Fig. 1)	Rectangular[1] element (4 × 4)	Triangular element (2 × 2)	Triangular element (3 × 3)	Rectangular element (3 × 3)	Rectangular element (4 × 4)	Series solution[9]
0.5	A	10,928	10,650	10,632	10,820	10,732	10,077
0.5	B	30,580	27,871	27,454	29,292	28,423	...
0.5	C	28,699	26,331	26,006	27,561	26,840	
0.7	A	8,021	7,982	7,966	8,105	8,038	7,930
0.7	B	20,878	20,067	19,622	21,138	20,396	19,074
0.7	C	17,763	17,217	16,914	17,945	17,493	15,704
1.0	A	7,222	7,199	7,161	7,290	7,225	7,091
1.0	B	17,211	17,199	16,595	18,225	17,386	16,235
1.0	C	11,851	11,917	11,609	12,262	11,916	11,012

Table 3 Critical stresses for 7-layer uniaxially loaded plate with different properties in different layers

Aspect ratio	Triangular element (2 × 2)	Rectangular element (4 × 4)	Rectangular element[1] (4 × 4)	Rectangular element[1] (8 × 8)	Series solution*[8]
0.4	14,796	14,890	50,000	75,000	11,886
0.7	9,831	9,893	...	...	8,712
1.0	9,213	9,270	...	...	8,443
1.2	9,699	9,761	33,529	39,081	8,985
1.6	11,830	11,915	...	...	11,085
2.0	15,069	15,184	31,982	37,576	14,194

three-layered rectangular sandwich plates with different boundary conditions and for different aspect ratios were solved for uniform uniaxial compression using both triangular and rectangular elements. It can be concluded, based on the results given in Table 1, that both rectangular elements give results of comparable accuracy, while the triangular element is definitely superior to the two rectangular elements. These results are also compared with the existing analytical solution in the same tables. The accuracy and convergence are seen to be excellent. The plate dimensions and properties are $B = 23.5$ in., A is defined by aspect ratio. Aluminum stiff layers: $t = 0.021$ in., $E = 9.5 \times 10^6$ psi, $v = 0.25$. Balsa core layer: $h = 0.181$ in., $G = 19{,}000$ psi.

2) The series solution available in literature (Wong and Salama[8]) and finite element (rectangular hybrid) solution (Lundgren and Salama[1]) of multilayer sandwich plates are based on average properties of individual layer. The reliability of this method is doubtful because of the variation of properties from layer to layer[3]; the present method is quite capable of taking into account such variations.[4] However, for multilayer sandwich plates with identical properties in all layers, the results should agree with those obtained by hybrid element[1] or series solutions technique, and a comparison in Table 2 shows a close agreement in the results. The plate dimensions and properties are $B = 100$ in., A is defined by aspect ratios. Stiff layers: $E = 30 \times 10^6$ psi; a) five-layer case $t = 0.025$ in., $v = 0.250$; b) seven-layer case $t = 0.020$ in., $v = 0.300$. Core layer: $G = 10^4$ psi; a) five-layer case $h = 0.300$ in.; b) seven-layer case $h = 0.200$ in.

3) The existing solutions based on average properties will differ from the results obtained from present theory, whenever there is variation in properties across the thickness. Table 3 presents results for simply supported, uniaxially loaded rectangular plate with different aspect ratios. It is seen that present solutions using triangular and rectangular elements are very similar to one another; however, the series solutions give consistently lower values compared to those obtained by present theory. The results obtained by Lundgren and Salama[1] are incorrect because they are much higher than those obtainable for a solid plate of equal thickness. The plate dimensions are $B = 100$ in., A is defined by aspect ratios, $v = 0.3$

$$E_{x1} = E_{y1} = E_{x4} = E_{y4} = 30 \times 10^6 \text{ psi}$$
$$E_{x2} = E_{y2} = E_{x3} = E_{y3} = 10^7 \text{ psi}$$
$$t_1 = 0.5 \text{ in.}, \quad t_2 = 0.075 \text{ in.}, \quad t_3 = 0.1 \text{ in.}, \quad t_4 = 0.06 \text{ in.}$$
$$h_1 = h_3 = 0.2 \text{ in.}, \quad h_2 = 0.25 \text{ in.}, \quad G_{xz1} = G_{yz1} = 10^4 \text{ psi}$$
$$G_{xz2} = G_{yz2} = 8000 \text{ psi}, \quad G_{xz3} = G_{yz3} = 12{,}000 \text{ psi}$$

The solution for a five-layer, simply supported square plate of 100 in. side with a circular hole of radius 16.33 in. (material properties of the plate are given in Sec. B) with uniform biaxial loading of equal magnitude in both directions, is now presented in order to show the extent of adaptability of triangular element in dealing with arbitrary shapes of plate geometry. The critical stress is found to be 3268 psi and for that without the central hole it is 4050 psi.

Table 2 Critical stresses for 5- and 7-layered uniaxially loaded plates

Aspect ratio	Triangular element (2 × 2) 5-Layer	Triangular element (2 × 2) 7-Layer	Rectangular element (4 × 4) 5-Layer	Rectangular element (4 × 4) 7-Layer	Series solution[8] 5-Layer	Series solution[8] 7-Layer
0.4	16,003	14,301	16,128	14,414	15,566	13,908
0.7	9,428	8,381	9,493	8,438	9,291	8,260
1.0	8,523	7,566	8,578	7,613	8,427	7,481
1.2	8,873	7,873	8,931	7,923	8,781	7,792
1.6	10,701	9,491	10,778	9,558	10,600	9,402
2.0	13,562	12,027	13,662	12,114	13,433	11,912

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